ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 333 (2007) 667–678

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Positive periodic solutions of nonautonomous functional differential systems

Na Zhang [a], Binxiang Dai [b,*,1], Yuming Chen [c,2]

[a] *Department of Mathematics, College of Science, Shenzhen University, Shenzhen, Guangdong 518060, China*

[b] *School of Mathematical Science and Computing Technology, Central South University, Changsha, Hunan 410075, China*

[c] *Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario, N2L 3C5, Canada*

Received 19 March 2006

Available online 29 December 2006

Submitted by T. Krisztin

Abstract

Considered is the periodic functional differential system with a parameter, $x'(t) = A(t, x(t))x(t) + \lambda f(t, x_t)$. Using the eigenvalue problems of completely continuous operators, we establish some criteria on the existence of positive periodic solutions. Moreover, we apply the results to a couple of population models and obtain sufficient conditions for the existence of positive periodic solutions, which are compared with existing ones.

Keywords: Functional differential system; Positive periodic solution; Eigenvalue; Cone; Population model

1. Introduction

One characteristic phenomenon of population dynamics is the often observed oscillatory behavior of the population densities. To better understand such a phenomenon, one mechanism is

* Corresponding author.

E-mail addresses: zn19791020@tom.com (N. Zhang), bxdai@hnu.cn (B. Dai), ychen@wlu.ca (Y. Chen).

1 Research was supported by the National Natural Science Foundation of China (10471153) and Hunan Provincial Natural Science Foundation of China (06JJ20006).

2 Research was partially supported by the Natural Science and Engineering Research Council of Canada (NSERC) and the Early Research Award Program (ERA) of Ontario.

doi:10.1016/j.jmaa.2006.09.084

to introduce time delays in the models, which results in models described by functional differential equations. As pointed out by Li and Kuang [6], more realistic and interesting models of single or multiple species growth should take into account both the seasonality of the changing environment (especially the periodically changing environment) and the effects of time delays. This motivates the study of periodic functional differential equations.

One important question is whether these periodic functional differential equations can support positive periodic solutions. Such question has been studied extensively in the literature. See, for example, [1,4,5,7–16] and the references therein.

In the above-mentioned references, most of the results on the existence of positive solutions are established by applying fixed point theorems such as the Krasnosel'skii fixed point theorem [1,5,9,12,14] and the generalized form of Leggett–Williams fixed point theorem [7,11]. Recently, based on the eigenvalue problems of completely continuous operators, Liu and Li [10] obtained some criteria on existence of positive periodic solutions of the following functional differential equation with parameter

$$y'(t) = -a(t)y(t) + \lambda h(t) f\big(y\big(t - \tau(t)\big)\big).$$

On the other hand, most of the equations are scalar delay differential equations with discrete delays and they can be written as

$$x'(t) = \pm a(t)x(t) \mp \lambda h(t) f\big(x\big(t - \tau(t)\big)\big).$$

Recently, Wang [14] considered the following more general scalar periodic equation

$$x'(t) = a(t)g\big(x(t)\big)x(t) - \lambda b(t) f\big(x\big(t - \tau(t)\big)\big).$$

Motivated by the above considerations, in this paper, we study the following n-dimensional functional differential system

$$x'(t) = A\big(t, x(t)\big)x(t) + \lambda f(t, x_t), \tag{1.1}$$

where $\lambda > 0$ is a parameter; $A(t, x(t)) = \operatorname{diag}[a_1(t, x(t)), \dots, a_n(t, x(t))]$ such that $a_i(t, x(t))$ is continuous and $a_i(t + \omega, \xi) = a_i(t, \xi)$ $(i = 1, \dots, n)$. Denote by BC the Banach space of bounded continuous functions $\phi = (\phi_1, \dots, \phi_n) : \mathbb{R} \to \mathbb{R}^n$ equipped with the norm $\|\phi\| = \sup_{\theta \in \mathbb{R}} \sum_{i=1}^n |\phi_i(\theta)|$. For $x \in BC$ and $t \in \mathbb{R}$, $x_t \in BC$ is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in \mathbb{R}$. The functional $f(t, x_t)$ defined on $\mathbb{R} \times BC$ is ω-periodic whenever x is ω-periodic, and $\omega > 0$ is a constant.

System (1.1) has been extensively investigated in the literature as models for bio-mathematics, neural networks and population dynamics. It includes the system of Volterra integro-differential equations [4]

$$\dot{x}_i(t) = x_i(t)\left[a_i(t) - \lambda \sum_{j=1}^n \left(b_{ij}(t)x_j(t) + \int_{-\infty}^{t} C_{ij}(t, s) g_{ij}\big(x_j(s)\big)\, \mathrm{d}s\right)\right], \tag{1.2}$$

which governs the population growth of interacting species $x_i(t)$, $i = 1, \dots, n$. It also includes the generalized n-species Gilpin–Ayala competition model with distributed infinite delay

$$\dot{x}_i(t) = x_i(t)\left[r_i(t) - \lambda \sum_{j=1}^n \int_{-\infty}^{0} \big(x_j(t + s)\big)^{\theta_{ij}}\, \mathrm{d}\mu_{ij}(t, s)\right], \quad i = 1, \dots, n, \tag{1.3}$$

where $\theta_{ij} \in (0, \infty)$, $r_i \in C(\mathbb{R}, [0, \infty))$ is ω-periodic, $\mu_{ij}(t, s)$ is continuous and ω-periodic in t, and is nondecreasing, bounded and continuous from the left hand in s such that

$\int_{-\infty}^{0} \mathrm{d}\mu_{ij}(t,s) < \infty$, $i, j = 1, \dots, n$. For more details about the background of the model, see Fan and Wang [2].

The purpose of this paper is to study the existence of positive periodic solutions to system (1.1). Our approach is the same as that used by Liu and Li [10], where they studied the scalar equation

$$\dot{y}(t) = -a(t)y(t) + \lambda h(t) f\big(y\big(t - \tau(t)\big)\big).$$

Precisely, in Section 2, we establish some results on the existence of positive periodic solutions of system (1.1) by employing the eigenvalue problems of completely continuous operators. Then, in Section 3, we apply these results to system (1.2) and system (1.3) and compare the obtained results with existing ones.

We shall conclude this section with some notations. Denote $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_- = (-\infty, 0]$, and $BC_+^n = \{\phi \in BC\colon\ \phi(t) \in \mathbb{R}_+^n \text{ for } t \in \mathbb{R}\}$. For $\mathbb{R}^n$, the norm $|\cdot|$ on it is defined by $|x| = \sum_{i=1}^{n} |x_i|$, where $x = (x_1, \dots, x_n)$. Finally, for two $m \times n$ matrices A and B, $A \geqslant B$ ($A < B$) means that the inequality is satisfied entrywisely. In particular, A is said to be a nonnegative matrix if $A \geqslant 0$.

2. Main results

In this section, we make the following assumptions:

(H1) There exist $B(t) = \mathrm{diag}[b_1(t), \dots, b_n(t)]$ and $C(t) = \mathrm{diag}[c_1(t), \dots, c_n(t)]$, where $b_i(t), c_i(t) \in C(\mathbb{R}, \mathbb{R}_+)$ are ω-periodic and $\int_0^{\omega} b_i(t)\,\mathrm{d}t > 0$, $i = 1, \dots, n$, such that

$$B(t) \leqslant A\big(t, \varphi(t)\big) \leqslant C(t) \quad \text{for all } (t, \varphi) \in \mathbb{R} \times BC_+^n.$$

(H2) $f(t, 0) = 0$ for all $t \in \mathbb{R}$.
(H3) $f(t, \varphi_t) \leqslant 0$ for all $(t, \varphi) \in \mathbb{R} \times BC_+^n$.
(H4) $f(t, \varphi_t)$ is a continuous function of t for each $\varphi \in BC_+^n$.
(H5) For any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for $\phi, \psi \in BC_+^n$, $\|\phi\| \leqslant L$, $\|\psi\| \leqslant L$ and $\|\phi - \psi\| \leqslant \delta$, $t \in [0, \omega]$,

$$\big|f(t, \phi_t) - f(t, \psi_t)\big| < \varepsilon.$$

For the convenience of the readers, we cite some pertinent definitions and lemmas.

Definition 2.1. [3] Let X be a Banach space and P be a closed, nonempty subset of X. P is a (convex) cone if

(i) $x, y \in P$ and $\alpha, \beta \in \mathbb{R}_+$ imply $\alpha x + \beta y \in P$.
(ii) $x \in P$ and $-x \in P$ imply $x = 0$.

Every cone $P \subset X$ induces a partial ordering in X. We define "$\leqslant$" with respect to P by $x \leqslant y$ if and only if $y - x \in P$.

Definition 2.2. [3] Let X be a Banach space and $D \subset X$, $0 \in D$. The operator $L : D \to X$ is such that $L0 = 0$. $x_\lambda \neq 0$ is said to be an eigenvector of the eigenvalue λ of L if $Lx_\lambda = \lambda x_\lambda$.

Lemma 2.1. [3] *Suppose D is an open subset of an infinite-dimensional real Banach space X, $0 \in D$, and P is a cone of X. If the operator $\Gamma : P \cap \overline{D} \to P$ is completely continuous with $\Gamma 0 = 0$ and satisfies $\inf_{x \in P \cap \partial D} \|\Gamma x\| > 0$, then Γ has an eigenvector on $P \cap \partial D$ associated with a positive eigenvalue. That is, there exist $x_0 \in P \cap \partial D$ and $\mu_0 > 0$ such that $\Gamma x_0 = \mu_0 x_0$.*

To study system (1.1), we introduce

$$X = \{x \in C(\mathbb{R}, \mathbb{R}^n):\ x(t+\omega) = x(t) \text{ for } t \in \mathbb{R}\},$$

endowed with the usual linear structure as well as the norm

$$\|x\| = \sum_{i=1}^{n} |x_i|_0 \quad \text{for } x = (x_1, \dots, x_n) \in X,$$

where

$$|x_i|_0 = \sup_{t \in [0,\omega]} |x_i(t)|, \quad i = 1, \dots, n.$$

Then X is a Banach space. If $x \in X$ is a solution of system (1.1), then, for $i = 1, \dots, n$,

$$\left[x_i(t)\exp\left(-\int_0^t a_i(s, x(s))\,\mathrm{d}s\right)\right]' = \lambda \exp\left(-\int_0^t a_i(s, x(s))\,\mathrm{d}s\right) f_i(t, x_t). \tag{2.1}$$

Integrating both sides of (2.1) over $[t, t+\omega]$, we obtain

$$x_i(t) = \lambda \int_t^{t+\omega} G_x^i(t,s) f_i(s, x_s)\,\mathrm{d}s, \quad i = 1, \dots, n,$$

where

$$G_x^i(t,s) = \frac{\exp\left(-\int_t^s a_i(\theta, x(\theta))\,\mathrm{d}\theta\right)}{\exp\left(-\int_0^\omega a_i(\theta, x(\theta))\,\mathrm{d}\theta\right) - 1}, \quad i = 1, \dots, n.$$

Let

$$\sigma = \min_{1 \leqslant i \leqslant n} \frac{\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)\left[1 - \exp\left(-\int_0^\omega b_i(\theta)\,\mathrm{d}\theta\right)\right]}{1 - \exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}.$$

From the assumption (H1), it can be easily obtained that $\sigma < 1$. Then we define two cones in X as follows:

$$P_1 = \{x \in X:\ x_i(t) \geqslant \sigma |x_i|_0 \text{ for } t \in \mathbb{R} \text{ and } i = 1, \dots, n\};$$
$$P_2 = \{x \in X:\ x(t) \geqslant 0 \text{ for } t \in \mathbb{R}\}.$$

Meanwhile, we define an operator Φ on X by

$$(\Phi x)(t) = \big((\Phi_1 x)(t), (\Phi_2 x)(t), \dots, (\Phi_n x)(t)\big),$$

where

$$(\Phi_i x)(t) = \int_t^{t+\omega} G_x^i(t,s) f_i(s, x_s)\,\mathrm{d}s, \quad i = 1, \dots, n. \tag{2.2}$$

Noting that $G_x^i(t,s) = G_x^i(t+\omega, s+\omega)$, it is obvious that Φ is well defined, i.e., $\Phi x \in X$ for $x \in X$. In addition, it is easy to check that $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t)) \geqslant 0$ is a positive ω-periodic solution of system (1.1) associated with λ^* if and only if $x^* \in P_2$ is an eigenvector of the operator Φ associated with the eigenvalue $\frac{1}{\lambda^*} > 0$, that is, $\Phi x^* = \frac{1}{\lambda^*} x^*$.

Lemma 2.2. *The mapping Φ maps P_1 into P_1, i.e., $\Phi P_1 \subset P_1$.*

Proof. By assumption (H1), we can easily see that, for $i = 1, \ldots, n$, and $t \leqslant s \leqslant t+\omega$,

$$\frac{1}{\exp\left(-\int_0^\omega b_i(\theta)\,\mathrm{d}\theta\right) - 1} \leqslant G_x^i(t,s) \leqslant \frac{\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}{\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right) - 1}. \tag{2.3}$$

Then (2.2) combined with (2.3) gives us

$$\begin{aligned}
\left|(\Phi_i x)(t)\right| &\leqslant \int_t^{t+\omega} \frac{1}{1 - \exp\left(-\int_0^\omega b_i(\theta)\,\mathrm{d}\theta\right)} \left|f_i(s, x_s)\right| \mathrm{d}s \\
&\leqslant \frac{1}{1 - \exp\left(-\int_0^\omega b_i(\theta)\,\mathrm{d}\theta\right)} \int_0^\omega \left|f_i(s, x_s)\right| \mathrm{d}s.
\end{aligned}$$

It follows that

$$\left|(\Phi_i x)\right|_0 \leqslant \frac{1}{1 - \exp\left(-\int_0^\omega b_i(\theta)\,\mathrm{d}\theta\right)} \int_0^\omega \left|f_i(s, x_s)\right| \mathrm{d}s,$$

or

$$\int_0^\omega \left|f_i(s, x_s)\right| \mathrm{d}s \geqslant \left[1 - \exp\left(-\int_0^\omega b_i(\theta)\,\mathrm{d}\theta\right)\right] \left|(\Phi_i x)\right|_0.$$

Therefore,

$$\begin{aligned}
(\Phi_i x)(t) &\geqslant \frac{\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}{1 - \exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)} \int_0^\omega \left|f_i(s, x_s)\right| \mathrm{d}s \\
&\geqslant \frac{\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)\left[1 - \exp\left(-\int_0^\omega b_i(\theta)\,\mathrm{d}\theta\right)\right]}{1 - \exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)} \left|(\Phi_i x)\right|_0 \\
&\geqslant \sigma \left|(\Phi_i x)\right|_0,
\end{aligned}$$

which means that $\Phi x \in P_1$. This completes the proof. □

Lemma 2.3. *The operator $\Phi : P_2 \to X$ is completely continuous.*

Proof. Under assumptions (H4) and (H5), it is clear that the operator Φ is continuous. Next, we show that Φ is compact.

Let $S \subseteq P_2$ be any bounded set. Then, by the assumption (H5), there exists a constant $M > 0$ such that

$$\left|f_i(t, x_t)\right| \leqslant M, \quad \text{for } (t, x) \in [0, \omega] \times S,\ i = 1, 2, \ldots, n.$$

From (2.2), we have

$$|\Phi_i x|_0 \leqslant \frac{1}{1-\exp\bigl(-\int_0^\omega b_i(\theta)\,\mathrm{d}\theta\bigr)} M\omega.$$

This yields

$$\|\Phi x\| = \sum_{i=1}^{n} |\Phi_i x|_0 \leqslant M\omega \sum_{i=1}^{n} \frac{1}{1-\exp\bigl(-\int_0^\omega b_i(\theta)\,\mathrm{d}\theta\bigr)}.$$

In view of the definition of Φ, we have

$$\begin{aligned}(\Phi_i x)'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_t^{t+\omega} G_x^i(t,s) f_i(s,x_s)\,\mathrm{d}s\right)\\ &= a_i\bigl(t,x(t)\bigr)\Phi_i x + f_i(t,x_t), \quad i=1,2,\ldots,n.\end{aligned}$$

So we obtain

$$\begin{aligned}\bigl|(\Phi_i x)'(t)\bigr| &\leqslant \bigl|a_i\bigl(t,x(t)\bigr)\bigr||\Phi_i x|_0 + \bigl|f_i(t,x_t)\bigr|\\ &\leqslant M(\tilde{C}\tilde{B}\omega + 1),\end{aligned}$$

where

$$\tilde{C} = \max_{1\leqslant i\leqslant n, t\in[0,\omega]} c_i(t), \qquad \tilde{B} = \max_{1\leqslant i\leqslant n} \frac{1}{1-\exp\bigl(-\int_0^\omega b_i(\theta)\,\mathrm{d}\theta\bigr)}.$$

Hence, $\Phi S \subseteq X$ is a family of uniformly bounded and equi-continuous functions. By the Ascoli–Arzela theorem [17], Φ is a compact operator. So Φ is completely continuous. This completes the proof. □

Before stating our results, we introduce the following two notations:

$$f_0 = \lim_{\substack{\phi\in P_1\\ \|\phi\|\to 0}} \frac{\int_0^\omega |f(s,\phi_s)|\,\mathrm{d}s}{\|\phi\|}, \qquad f_\infty = \lim_{\substack{\phi\in P_1\\ \|\phi\|\to \infty}} \frac{\int_0^\omega |f(s,\phi_s)|\,\mathrm{d}s}{\|\phi\|}.$$

Also, define, for r a positive number, Ω_r by

$$\Omega_r = \bigl\{x\in X\colon\ \|x\| < r\bigr\}.$$

Theorem 2.1. *Assume* (H1)–(H5) *hold and* $0 < f_\infty < \infty$. *Then there exist positive constants* R_0, λ_1 *and* λ_2 *with* $\lambda_1 < \lambda_2$ *such that, for any* $r > R_0$, *system* (1.1) *has a positive* ω*-periodic solution* $x^r(t)$ *associated with some* $\lambda_r \in [\lambda_1, \lambda_2]$ *and* $\|x^r\| = r$.

Proof. Since $0 < f_\infty < +\infty$, there exist $\varepsilon_2 > \varepsilon_1 > 0$ and $R_0 > 0$ such that

$$\varepsilon_1\|\phi\| < \int_0^\omega \bigl|f(s,\phi_s)\bigr|\,\mathrm{d}s < \varepsilon_2\|\phi\| \quad \text{for } \|\phi\| \geqslant R_0,\ \phi\in P_1. \tag{2.4}$$

Suppose $r > R_0$, then Ω_r is a bounded open subset of X and $0 \in \Omega_r$. For $x \in P_1 \cap \partial\Omega_r$, we have

$$
\begin{aligned}
\|\Phi x\| &= \sum_{i=1}^{n} \max_{t\in[0,\omega]} \left|(\Phi_i x)(t)\right| \\
&\geqslant \sum_{i=1}^{n} \left|(\Phi_i x)(t)\right| \\
&= \sum_{i=1}^{n} \int_t^{t+\omega} G_x^i(t,s) f_i(s,x_s)\,\mathrm{d}s \\
&\geqslant \sum_{i=1}^{n} \frac{\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}{1-\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)} \int_0^\omega \left|f_i(s,x_s)\right|\mathrm{d}s \\
&\geqslant \min_{1\leqslant i\leqslant n}\left\{\frac{\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}{1-\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}\right\} \int_0^\omega \sum_{i=1}^{n}\left|f_i(s,x_s)\right|\mathrm{d}s \\
&\geqslant \min_{1\leqslant i\leqslant n}\left\{\frac{\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}{1-\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}\right\}\varepsilon_1 r > 0.
\end{aligned}
$$

It follows that

$$
\inf_{x\in P_1\cap\partial\Omega_r} \|\Phi x\| \geqslant \min_{1\leqslant i\leqslant n}\left\{\frac{\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}{1-\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}\right\}\varepsilon_1 r > 0.
$$

Moreover, Φ is completely continuous with $\Phi 0 = 0$. It follows from Lemma 2.1 that the operator Φ has an eigenvector $x^r \in P_1$ associated with the eigenvalue $\mu_r > 0$ such that $\|x^r\| = r$. Set $\lambda_r = \frac{1}{\mu_r}$. Then x^r is a positive ω-periodic solution of system (1.1). We determine λ_1 and λ_2 as follows. From

$$
\begin{aligned}
\left(x^r\right)_i(t) &= \lambda_r \int_t^{t+\omega} G_{x^r}^i(t,s) f_i\left(s,x_s^r\right)\mathrm{d}s \\
&\leqslant \lambda_r \frac{1}{1-\exp\left(-\int_0^\omega b_i(\theta)\,\mathrm{d}\theta\right)} \int_0^\omega \left|f_i\left(s,x_s^r\right)\right|\mathrm{d}s \\
&\leqslant \lambda_r \frac{1}{1-\exp\left(-\int_0^\omega b_i(\theta)\,\mathrm{d}\theta\right)}\varepsilon_2 r, \quad i=1,\dots,n,
\end{aligned}
$$

and $\|x^r\| = r$ we can get

$$
\lambda_r \geqslant \frac{1}{\varepsilon_2 \sum_{i=1}^{n} \frac{1}{1-\exp\left(-\int_0^\omega b_i(\theta)\,\mathrm{d}\theta\right)}} =: \lambda_1.
$$

On the other hand,

$$
\left(x^r\right)_i(t) \geqslant \lambda_r \frac{\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}{1-\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)} \int_0^\omega \left|f_i\left(s,x_s^r\right)\right|\mathrm{d}s, \quad i=1,\dots,n.
$$

It follows from

$$\begin{aligned}\|x^r\| = r &\geqslant \lambda_r \min_{1\leqslant i\leqslant n}\left\{\frac{\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}{1-\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}\right\}\int_0^\omega \left|f\left(s,x_s^r\right)\right|\mathrm{d}s\\
&\geqslant \lambda_r \min_{1\leqslant i\leqslant n}\left\{\frac{\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}{1-\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}\right\}\varepsilon_1 r\end{aligned}$$

that

$$\lambda_r \leqslant \max_{1\leqslant i\leqslant n}\left\{\frac{1-\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}{\varepsilon_1\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}\right\} =: \lambda_2.$$

In summary, $\lambda_r \in [\lambda_1, \lambda_2]$ and this completes the proof. □

Remark 2.1. From the proof of Theorem 2.1, we see that the condition $0 < f_\infty < \infty$ can be relaxed to (2.4). For the simplicity of presentation, we state Theorem 2.1 that way. Similar remarks can be made for the following results.

Theorem 2.2. *Assume* (H1)–(H5) *hold and* $0 < f_0 < \infty$. *Then there exist positive constants* $r_0 > 0$, $\tilde{\lambda}_1$ *and* $\tilde{\lambda}_2$ *with* $\tilde{\lambda}_1 < \tilde{\lambda}_2$ *such that, for any* $0 < r < r_0$, *system* (1.1) *has a positive* ω*-periodic solution* $\tilde{x}^r(t)$ *associated with some* $\tilde{\lambda}_r \in [\tilde{\lambda}_1, \tilde{\lambda}_2]$ *and* $\|\tilde{x}^r\| = r$.

Proof. Since $0 < f_0 < +\infty$, there exist $0 < l_1 < l_2$ and $r_0 > 0$ such that

$$l_1\|\phi\| < \int_0^\omega \left|f(s,\phi_s)\right|\mathrm{d}s < l_2\|\phi\| \quad \text{for } 0<\|\phi\|<r_0,\ \phi\in P_1.$$

For $r \in (0, r_0)$, Ω_r is a bounded open subset of X and $0 \in \Omega_r$. Moreover, for $x \in P_1 \cap \partial\Omega_r$,

$$\begin{aligned}\|\Phi x\| &\geqslant \sum_{i=1}^n \left|(\Phi x)_i(t)\right|\\
&= \sum_{i=1}^n \int_t^{t+\omega} G_x^i(t,s) f_i(s,x_s)\,\mathrm{d}s\\
&\geqslant \min_{1\leqslant i\leqslant n}\left\{\frac{\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}{1-\exp\left(-\int_0^\omega c_i(\theta)\,\mathrm{d}\theta\right)}\right\} l_1 r > 0,\end{aligned}$$

which implies that $\inf_{x\in P_1\cap\partial\Omega_r}\|\Phi x\| > 0$. The remaining proof is quite similar to that of Theorem 2.1 and hence is omitted. This completes the proof. □

The following two results can be established using similar arguments as those for Theorem 2.1 and Theorem 2.2, respectively.

Theorem 2.3. *Assume* (H1)–(H5) *hold and* $f_\infty = \infty$. *Then there exist positive constants* $\bar{R}_0$ *and* $\bar{\lambda}$ *such that, for any* $r > \bar{R}_0$, *system* (1.1) *has a positive* ω*-periodic solution* $\bar{x}^r(t)$ *associated with some* $\bar{\lambda}_r \leqslant \bar{\lambda}$*and* $\|\bar{x}^r\| = r$.

Theorem 2.4. *Assume* (H1)–(H5) *hold and* $f_0 = \infty$. *Then there exist positive constants* $\hat{r}_0$ *and* $\hat{\lambda}$ *such that, for any* $0 < r < \hat{r}_0$, *system* (1.1) *has a positive* ω*-periodic solution* $\hat{x}^r(t)$ *associated with some* $\hat{\lambda}_r \leqslant \hat{\lambda}$ *and* $\|\hat{x}^r\| = r$.

Remark 2.2. By similar arguments, Theorems 2.1–2.4 also hold for the following system

$$x'(t) = -A\big(t, x(t)\big)x(t) - \lambda f(t, x_t),$$

where all the variables and parameters are same as those in system (1.1).

3. Some applications

In this section, we apply our main results obtained in Section 2 to a couple of well-known models in population dynamics.

First, we consider the following Volterra integro-differential equations

$$\dot{x}_i(t) = x_i(t)\left[a_i(t) - \lambda \sum_{j=1}^{n} \left(b_{ij}(t)x_j(t) + \int_{-\infty}^{t} C_{ij}(t,s)g_{ij}\big(x_j(s)\big)\,\mathrm{d}s\right)\right],$$
$$i = 1, \ldots, n, \tag{3.1}$$

where $a_i, b_{ij} \in C(\mathbb{R}, \mathbb{R}_+)$ are ω-periodic functions with $\int_0^\omega a_i(t)\,\mathrm{d}t > 0$ and $\int_0^\omega b_{ii}(t)\,\mathrm{d}t > 0$; $C_{ij}(t,s) \geqslant 0$ and $C_{ij}(t+\omega, s+\omega) = C_{ij}(t,s)$ for all $(t,s) \in \mathbb{R}^2$; $g_{ij} \in C(\mathbb{R}_+, \mathbb{R}_+)$, $i, j = 1, \ldots, n$.

Theorem 3.1. *Suppose that* $\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} C_{ij}(t,s)\,\mathrm{d}s < \infty$. *Then there exist positive constants* R_0 *and* λ_0 *such that, for any* $r > R_0$, *system* (3.1) *has a positive* ω*-periodic solution* $x^r(t)$ *associated with* $\lambda_r \leqslant \lambda_0$ *and* $\|x^r\| = r$.

Proof. Note that system (3.1) is a special form of system (1.1) with

$$A\big(t, x(t)\big) = \operatorname{diag}\big[a_1(t), \ldots, a_n(t)\big]$$

and $f = (f_1, \ldots, f_n)$, where

$$f_i(t, x_t) = -x_i(t) \sum_{j=1}^{n} \left(b_{ij}(t)x_j(t) + \int_{-\infty}^{t} C_{ij}(t,s)g_{ij}\big(x_j(s)\big)\,\mathrm{d}s\right),$$

$i = 1, \ldots, n$, and (H1)–(H5) are satisfied. For $x \in P_1$ and $i = 1, \ldots, n$,

$$\begin{aligned}
\int_0^\omega \big|f_i(s, x_s)\big|\,\mathrm{d}s &= \sum_{j=1}^{n} \int_0^\omega x_i(s)\left(x_j(s)b_{ij}(s) + \int_{-\infty}^{s} C_{ij}(s,\theta)g_{ij}\big(x_j(\theta)\big)\,\mathrm{d}\theta\right)\mathrm{d}s \\
&\geqslant \sum_{j=1}^{n} \int_0^\omega x_i(s)x_j(s)b_{ij}(s)\,\mathrm{d}s \\
&\geqslant \int_0^\omega x_i^2(s)b_{ii}(s)\,\mathrm{d}s \\
&\geqslant \sigma^2 |x_i|_0^2 \int_0^\omega b_{ii}(s)\,\mathrm{d}s.
\end{aligned}$$

Thus

$$\begin{aligned}\int_0^{\omega}\big|f(s,x_s)\big|\,\mathrm{d}s &= \sum_{i=1}^{n}\int_0^{\omega}\big|f_i(s,x_s)\big|\,\mathrm{d}s\\ &\geqslant \sum_{i=1}^{n}\sigma^2|x_i|_0^2\int_0^{\omega}b_{ii}(s)\,\mathrm{d}s\\ &\geqslant \sigma^2\min_{1\leqslant i\leqslant n}\int_0^{\omega}b_{ii}(s)\,\mathrm{d}s\sum_{i=1}^{n}|x_i|_0^2\\ &\geqslant \frac{\sigma^2}{n}\|x\|^2\min_{1\leqslant i\leqslant n}\int_0^{\omega}b_{ii}(s)\,\mathrm{d}s.\end{aligned}$$

It follows that

$$\frac{\int_0^{\omega}|f(s,x_s)|\,\mathrm{d}s}{\|x\|}\to\infty \quad \text{as } \|x\|\to\infty.$$

Now the conclusion follows directly from Theorem 2.3 and this completes the proof. □

Remark 3.1. System (3.1) was studied by Jiang et al. [4]. Using the fixed point index on cones, the authors showed that system (3.1) has a positive ω-periodic solution for all $\lambda > 0$. However, one of the assumptions is that g_{ij}s are increasing with $g_{ij}(0) = 0$, which is not required in Theorem 3.1.

Next, we consider the generalized n-species Gilpin–Ayala competition model with distributed infinite delay,

$$\dot{x}_i(t) = x_i(t)\left[r_i(t) - \lambda\sum_{j=1}^{n}\int_{-\infty}^{0}\big(x_j(t+s)\big)^{\theta_{ij}}\,\mathrm{d}\mu_{ij}(t,s)\right], \quad i = 1,\dots,n, \tag{3.2}$$

where $\theta_{ij}\in(0,\infty)$, $r_i\in C(\mathbb{R},\mathbb{R}_+)$ is an ω-periodic function; $\mu_{ij}(t,s)$ is a continuous ω-periodic function with respect to t, and is nondecreasing, bounded and continuous from left hand in s with $\int_{-\infty}^{0}\mathrm{d}\mu_{ij}(t,s) < \infty$, $i, j = 1,\dots,n$.

Theorem 3.2. *Suppose that* $A_{ii} =: \int_0^{\omega}(\int_{-\infty}^{0}\mathrm{d}\mu_{ii}(t,s))\,\mathrm{d}t \in (0,\infty)$, $i = 1,\dots,n$. *Then there exist positive constants* $\tilde{R}_0$ *and* $\tilde{\lambda}_0$ *such that, for any* $r > \tilde{R}_0$, *system* (3.2) *has a positive ω-periodic solution* $\tilde{x}^r$ *associated with* $\tilde{\lambda}_r \leqslant \tilde{\lambda}_0$ *and* $\|\tilde{x}^r\| = r$.

Proof. Again, system (3.2) can be written in the form of system (1.1) with

$$A\big(t,x(t)\big) = \mathrm{diag}\big[r_1(t),\dots,r_n(t)\big]$$

and $f = (f_1,\dots,f_n)$, where

$$f_i(t,x_t) = -x_i(t)\sum_{j=1}^{n}\int_{-\infty}^{0}\big(x_j(t+s)\big)^{\theta_{ij}}\,\mathrm{d}\mu_{ij}(t,s), \quad i = 1,\dots,n.$$

Also note that assumptions (H1)–(H5) are satisfied.

Denote

$$A = \min_{1 \leqslant i \leqslant n} A_{ii}, \qquad \theta = \min_{1 \leqslant i \leqslant n} \theta_{ii}.$$

For any $x \in P_1$ and $i = 1, \ldots, n$,

$$\begin{aligned}
\int_0^\omega \left| f_i(t, x_t) \right| \mathrm{d}t &= \sum_{j=1}^n \int_0^\omega x_i(t) \left[\int_{-\infty}^0 \left(x_j(t+s) \right)^{\theta_{ij}} \mathrm{d}\mu_{ij}(t,s) \right] \mathrm{d}t \\
&\geqslant \sum_{j=1}^n \sigma^{1+\theta_{ij}} |x_i|_0 |x_j|_0^{\theta_{ij}} \int_0^\omega \left(\int_{-\infty}^0 \mathrm{d}\mu_{ij}(t,s) \right) \mathrm{d}t \\
&\geqslant \sigma^{1+\theta_{ii}} |x_i|_0^{1+\theta_{ii}} \int_0^\omega \left(\int_{-\infty}^0 \mathrm{d}\mu_{ii}(t,s) \right) \mathrm{d}t \\
&\geqslant A\sigma^{1+\theta} |x_i|_0^{1+\theta}.
\end{aligned}$$

It follows that

$$\int_0^\omega \left| f(t, x_t) \right| \mathrm{d}t \geqslant A\sigma^{1+\theta} \sum_{i=1}^n |x_i|_0^{1+\theta}$$

and

$$\begin{aligned}
\frac{\int_0^\omega |f(t, x_t)| \, \mathrm{d}t}{\|x\|} &\geqslant A\sigma^{1+\theta} \frac{\sum_{i=1}^n |x_i|_0^{1+\theta}}{\sum_{i=1}^n |x_i|_0} \\
&\geqslant A\sigma^{1+\theta} \frac{\max_{1 \leqslant i \leqslant n} |x_i|_0^{1+\theta}}{n \max_{1 \leqslant i \leqslant n} |x_i|_0} \\
&= \frac{A\sigma^{1+\theta}}{n} \max_{1 \leqslant i \leqslant n} |x_i|_0^\theta.
\end{aligned}$$

Thus

$$\frac{\int_0^\omega |f(t, x_t)| \, \mathrm{d}t}{\|x\|} \to \infty \quad \text{as } \|x\| \to \infty.$$

Again, the conclusion follows immediately from Theorem 2.3 and this completes the proof. □

Remark 3.2. Fan and Wang [2] considered the special case of system (3.2) with $\lambda = 1$. Based on the method of coincidence degree, sufficient conditions for the existence of positive ω-periodic solutions are established. These conditions involve existence of finite number of positive solutions to a system of algebraic equations and inequalities on the coefficients and parameters. Obviously, Theorem 3.2 can include the results of [2] partly and our assumptions here are much weaker.

References

[1] S.S. Cheng, G. Zhang, Existence of positive periodic solutions for non-autonomous functional differential equations, Electron. J. Differential Equations 2001 (59) (2001) 1–8.

[2] M. Fan, K. Wang, Global periodic solutions of a generalized n-species Gilpin–Ayala competition model, Comput. Math. Appl. 40 (2000) 1141–1151.

[3] D.J. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Notes and Reports in Mathematics in Science and Engineering, vol. 5, Academic Press Inc., Boston, MA, 1988, pp. 2–99.

[4] D. Jiang, D. O'Regan, R.P. Agarwal, X. Xu, On the number of positive periodic solutions of functional differential equations and population models, Math. Models Methods Appl. Sci. 15 (2005) 555–573.

[5] D. Jiang, J. Wei, B. Zhang, Positive periodic solutions of functional differential equations and population models, Electron. J. Differential Equations 2002 (71) (2002) 1–13.

[6] Y. Li, Y. Kuang, Periodic solutions of periodic delay Lotka–Volterra equations and systems, J. Math. Anal. Appl. 255 (2001) 260–280.

[7] Y. Li, L. Zhu, Positive periodic solutions of nonlinear functional differential equations, Appl. Math. Comput. 156 (2004) 329–339.

[8] P. Liu, Y. Li, Positive periodic solutions of infinite delay functional differential equations depending on a parameter, Appl. Math. Comput. 150 (2004) 159–168.

[9] P. Liu, Y. Li, L. Lu, Positive periodic solutions of infinite delay functional differential equations, Far East J. Appl. Math. 18 (2005) 175–183.

[10] X.L. Liu, W.T. Li, Existence and uniqueness of positive periodic solutions of functional differential equations, J. Math. Anal. Appl. 293 (2004) 28–39.

[11] X.L. Liu, G. Zhang, S.S. Cheng, Existence of triple positive periodic solutions of a functional differential equation depending on a parameter, Abstr. Appl. Anal. 2004 (2004) 897–905.

[12] Y.N. Raffoul, Positive periodic solutions of systems of functional differential equations, J. Korean Math. Soc. 42 (2005) 749–759.

[13] A. Wan, D. Jiang, X. Xu, A new existence theory for positive periodic solutions to functional differential equations, Comput. Math. Appl. 47 (2004) 1257–1262.

[14] H. Wang, Positive periodic solutions of functional differential equations, J. Differential Equations 202 (2004) 354–366.

[15] K. Wang, Periodic solutions of a class of differential equations with deviating arguments, Acta Math. Sinica 37 (1994) 409–413 (in Chinese).

[16] G. Zhang, S.S. Cheng, Positive periodic solutions of nonautonomous functional differential equations depending on a parameter, Abstr. Appl. Anal. 7 (2002) 279–286.

[17] H.L. Roydin, Real Analysis, Macmillan Publishing Company, New York, 1998, pp. 169–170.